Chapter 8 Additional Reading

I. Deriving the PDE Solution for Pricing a Long-Term Bond

Assume that the instantaneous interest rate r_t follows any stochastic differential given by equation (8.5). We showed in this case that the bond price $B(t,T)$ satisfied the partial differential equation (8.15). For convenience, further assume that the price of the bond at maturity is 1 unit, creating the boundary condition $B(T,T) = 1$. Our goal is to solve for the price $B(t,T)$ at any time $t \leq T$. First, we prove that the solution takes the form:

$$(1) \quad B(t,T) = E_{\mathbb{P}}\left[\exp\left(-\int_t^T r_s ds - \frac{1}{2}\int_t^T \Theta^2(s,r_s)ds - \int_t^T \Theta(s,r_s)dZ_s\right) | \mathcal{F}_t\right]$$

for $t \leq T$. To prove this result, define the function

$$(2) \quad f(x) = \exp\left(-\int_t^x r_s ds - \frac{1}{2}\int_t^x \Theta^2(s,r_s)ds - \int_t^x \Theta(s,r_s)dZ_s\right).$$

Now, use Itô's Lemma to calculate the differential of $F(x) = B(x,T)\, f(x)$:

$$(3) \quad dF = \frac{\partial F}{\partial B}dB + \frac{\partial F}{\partial f}df + \frac{\partial^2 F}{\partial B \partial f}dBdf = fdB + Bdf + dBdf.$$

Note that higher-order partial derivatives of F with respect to B and f are equal to zero. For convenience, define

$$g(\tau) = -\int_t^\tau r_s ds - \frac{1}{2}\int_t^\tau \Theta^2(s,r_s)ds - \int_t^\tau \Theta(s,r_s)dZ_s,$$

so that $f(\tau) = e^{g(\tau)}$. By another application of Ito's Lemma, we have:

(4) $df = \frac{\partial f}{\partial g} dg + \frac{1}{2}\frac{\partial^2 f}{\partial g^2}(dg)^2 = f\left(-r_\tau d\tau - \frac{1}{2}\Theta^2 d\tau - \Theta dZ_\tau\right) + \frac{1}{2} f\Theta^2 d\tau = -f r_\tau d\tau - f\Theta dZ_\tau.$

To calculate *dBdf*, only the term $b\frac{\partial B}{\partial r} dZ_\tau$ is needed from equation (8.6) to approximate *dB*, and only the term $-f\Theta dZ_\tau$ is needed from equation (4) to approximate *df*. All of the other terms are negligible in order to estimate *dBdf*, since their effect on *dBdf* will be small compared to *dτ*. By equations (8.6) and (4), we have

(5) $$dBdf = -f\Theta b \frac{\partial B}{\partial r}(dZ_\tau)^2 = -f\Theta b\frac{\partial B}{\partial r} d\tau.$$

Substituting equations (8.6), (4), and (5) into equation (3) and simplifying gives

$$dF = f\left[\frac{\partial B}{\partial \tau} + (a - \Theta b)\frac{\partial B}{\partial r} + \frac{b^2}{2}\frac{\partial^2 B}{\partial r^2} - rB\right] d\tau - Bf\Theta dZ_\tau + fb\frac{\partial B}{\partial r} dZ_{\tau.}$$

However, as we showed earlier (equation 8.15) the expression inside the square brackets equals zero. Thus the differential *dF* simplifies to:

$$dF = -Bf\Theta dZ_\tau + fb\frac{\partial B}{\partial r} dZ_\tau.$$

Integrating this equation in the variable *τ* from *t* to *T* and taking the expectation with respect to the filtration $\mathcal{F}_t$ at time *t*, we have:

(6) $$E_{\mathbb{P}}[F(T)|\mathcal{F}_t] - E_{\mathbb{P}}[F(t)|\mathcal{F}_t] = E_{\mathbb{P}}[\int_t^T (-Bf\Theta + f\sigma_r \frac{\partial B}{\partial r}) dZ_\tau |\mathcal{F}_t].$$

We see by equation (6.1) that the right side of equation (6) equals 0. Observe that $F(t) = B(t, T)$ and that $B(t, T)$ is a determined function with respect to the filtration $\mathcal{F}_t$. Thus, $E_{\mathbb{P}}[F(t)|\mathcal{F}_t] = B(t,T)$. Referring to equation (2), we see that the expression $E_{\mathbb{P}}[F(T)|\mathcal{F}_t]$ is precisely the right-hand side of equation (1), and so we have derived equation (1).

The Closed Form Solution for the Vasicek Bond Price with constant risk premium Θ

In this case, equation (1) becomes

$$(7) \qquad B(t,T) = E_{\mathbb{P}}\left[\exp\left(-\int_t^T r_s ds - \frac{1}{2}\int_t^T \Theta^2 ds - \int_t^T \Theta dZ_s\right)|\mathcal{F}_t\right]$$

$$= e^{-\frac{1}{2}\Theta^2(T-t)} E_{\mathbb{P}}\left[\exp\left(-\int_t^T r_s ds - \int_t^T \Theta dZ_s\right)|\mathcal{F}_t\right].$$

By replacing the time variable t with s, and replacing r_0 with r_t so that time t is regarded as the current time, the solution for r_s for $t \le s \le T$ in equation (8.1) can be expressed as

$$r_s = \mu_s + \int_0^{s-t} \sigma_r e^{\lambda(u-s+t)}\, d\tilde{Z}_u$$

where

$$\mu_s = (r_t - \mu)e^{-\lambda(s-t)} + \mu$$

is a deterministic function for any $s \ge t$ and $\tilde{Z}_u = Z_{u+t} - Z_t$. Note that $\tilde{Z}_u$ is standard Brownian motion in the variable u. In order to evaluate the expectation in equation (7), we need to find a different way to express the process $-\int_t^T r_s ds - \int_t^T \Theta dZ_s$. Observe that

$$\int_t^T r_s ds = \int_t^T \mu_s ds + \int_t^T \int_0^{s-t} \sigma_r e^{\lambda(u-s+t)}\, d\tilde{Z}_u ds.$$

Interchanging the order of integration in the double integral gives

$$\int_t^T r_s ds = \int_t^T \mu_s ds + \int_0^{T-t} \int_{u+t}^{T} \sigma_r e^{\lambda(u-s+t)} ds\, d\tilde{Z}_u$$

$$= \int_t^T \mu_s ds + \int_0^{T-t} \frac{\sigma_r}{\lambda}\left[1 - e^{\lambda(u-T+t)}\right] d\tilde{Z}_u.$$

Observe that

$$\int_t^T \Theta dZ_s = \int_0^{T-t} \Theta d\tilde{Z}_s.$$

Thus

$$-\int_t^T r_s ds - \int_t^T \Theta dZ_s = -\int_t^T \mu_s ds - \int_0^{T-t} \left(\frac{\sigma_r}{\lambda}\left[1 - e^{\lambda(u-T+t)}\right] + \Theta\right) d\tilde{Z}_u.$$

Since μ_s is a deterministic function, then the function $-\int_t^T \mu_s ds$ is deterministic, and by the theorem in Section 6.1.5.3 the function $-\int_0^{T-t} (\frac{\sigma_r}{\lambda}\left[1 - e^{\lambda(u-T+t)}\right] + \Theta) d\tilde{Z}_u$ has a normal distribution with expectation taken at time *t* (which corresponds to time 0 for $\tilde{Z}_u$) equal to zero. Thus $-\int_t^T r_s ds - \int_t^T \Theta dZ_s$ is a random variable with a normal distribution and expectation at time *t*:

$$E_{\mathbb{P}}\left[-\int_t^T r_s ds - \int_t^T \Theta dZ_s \,|\mathcal{F}_t\right] = -\int_t^T \mu_s ds = -\int_t^T \left[(r_t-\mu)e^{-\lambda(s-t)} + \mu\right]ds$$

$$= \frac{\mu - r_t}{\lambda}\left[1 - e^{-\lambda(T-t)}\right] - \mu(T-t).$$

Its variance with respect to the filtration $\mathcal{F}_t$ is:

$$Var\left[-\int_t^T r_s ds - \int_t^T \Theta dZ_s \,|\mathcal{F}_t\right] = E_{\mathbb{P}}\left[\{-\int_0^{T-t} (\frac{\sigma_r}{\lambda}\left[1 - e^{\lambda(u-T+t)}\right] + \Theta)d\tilde{Z}_u\}^2|\mathcal{F}_t\right].$$

By the Ito Isometry,

$$Var\left[-\int_t^T r_s ds - \int_t^T \Theta dZ_s|\mathcal{F}_t\right] = \int_0^{T-t} \{\frac{\sigma_r}{\lambda}\left[1 - e^{\lambda(u-T+t)}\right] + \Theta\}^2 \,du$$

Evaluating the integral above is a straightforward calculation. First expand the quadratic integrand. Next, integrate term by term. After some algebraic manipulation, the resulting variance can be expressed as:

$$\frac{1}{\lambda}\left(1 - e^{-\lambda(T-t)}\right)\left(-\frac{2\sigma_r\Theta}{\lambda} - \frac{\sigma_r^2}{\lambda^2}\right) + (T-t)\left(\Theta^2 + \frac{2\sigma_r\Theta}{\lambda} + \frac{\sigma_r^2}{\lambda^2}\right) - \frac{\sigma_r^2}{2\lambda^3}\left(1 - e^{-\lambda(T-t)}\right)^2.$$

By equation (2.26), and after some algebraic manipulation, the price of the bond is:

(8) $$B(t,T) = exp\left[\frac{1}{\lambda}\left(1 - e^{-\lambda(T-t)}\right)(r_\infty - r_t) - r_\infty(T-t) - \frac{\sigma_r^2}{4\lambda^3}\left(1 - e^{-\lambda(T-t)}\right)^2\right],$$

where

$$r_\infty = \mu - \frac{\sigma_r \Theta}{\lambda} - \frac{\sigma_r^2}{2\lambda^2}.$$

II. Additional Applications for Mean Reverting Processes

As we discussed earlier, mean reverting processes such as Ornstein-Uhlenbeck have a wide range of applications in finance. Among them, in addition to bond pricing and yield curve mechanics are stochastic volatility models, exchange rate and commodity price modeling and arbitrage portfolio dynamics. In each of these scenarios, the rate, return or value has a long-term mean or normal rate, to which randomly fluctuating short-term values tend to revert.

We have already discussed a small number of powerful applications for mean-reverting processes. No new pricing models will be derived in this section. Instead, this section merely seeks to introduce substantially different applications for mean reverting processes in quantitative finance. These applications are further developed in sources discussed later in this chapter. First, we will focus on the evolution of pairs trading arbitrage portfolio values as a simple example of an application of Ornstein-Uhlenbeck processes, one that is very different from the bond pricing examples above.

<u>Illustration: Pairs Trading and the Ornstein-Uhlenbeck Process</u>

In this illustration, we consider the application of the Ornstein-Uhlenbeck process to arbitrage portfolio values, which, in a perfectly efficient market, should always be zero. However, realistically, arbitrage portfolio values do frequently drift away from zero for at least short periods, creating what portfolio managers call *basis risk*. Basis risk is the risk that markets might move too slowly to profit from an apparent arbitrage, or that markets might move opposite to the arbitrageur's expectations, at least in the short run.

Pairs trading is a strategy intended to exploit short-term deviations from a long-run equilibrium pricing relationship between two securities. As with other arbitrage strategies, pairs trading involves the simultaneous purchase and sale of similar securities. Pairs trading typically involves taking offsetting positions two different stocks (perhaps options or index contracts) with strong returns correlations, one long and one short such that gains in one position are expected to more than offset losses in the other position. For example, one might purchase GM stock when it seems underpriced relative to Ford stock, which might be shorted. If the Canadian and U.S. currencies are expected to exhibit strong positive correlations over the long run, pairs trades might buy one currency when it devalues against the other that is simultaneously shorted. While these types of hedges are rarely perfect, diverse portfolios of pairs can be comprised. Sufficiently large and diverse portfolios of pairs can mitigate the risks of the overall portfolio of pairs. Sometimes this type of portfolio composition is referred to as a type of *statistical arbitrage*, more broadly defined as short-term mean-reversion strategies involving large numbers of securities and very short holding periods (paraphrased from Lo [2010]).

Illustration: Berkshire-Hathaway Class A and B Shares

Consider a situation where a company, Berkshire Hathaway, has two different classes of shares trading in the market. As of December, 2011, Class A shares had a claim on Berkshire-Hathaway dividends that was 1,500 times higher than those of Class B shares. A very simple model might predict that Class A shares would sell at a price that is 1,500 times as high as the price of Class B shares. However, voting rights for Class A shares are 10,000 times as high as for Class B shares. These voting rights differentials and transactions costs might lead the two classes of shares to differ from the 1,500 to 1 ratio, at least for short periods of time. Nevertheless, we might expect that, at least in general, these two classes of shares experience similar proportional changes in price. Based on dividends alone, as long as voting rights are insignificant (it is reasonable to surmise that Warren Buffet probably has the bulk of the influence in Berkshire-Hathaway corporate elections), the 1,500 to 1 ratio should be a good approximation for the price differential.

One simple pairs trading strategy might involve taking a long position in Class A shares of Berkshire-Hathaway stock along with a short position in 1,500 times as many Class B shares when the Class A shares seem undervalued relative to the Class B shares. Pairs trading is essentially an arbitrage strategy anticipating that the deviation of a recent pricing relation between two securities is only temporary. Pairs traders typically focus either on the ratio between prices of two securities or the difference between their prices. When differences do arise, or when price ratios deviate from their norms, they sometimes take time to resolve. That is, notice on Table 1 the tendency for differences to tend to resolve to their mean value. Larger differences tend to take more time to revert to their means.

Date	**A-Adj Close**	**B-Adj Close**	**Diff.**	**Date**	**A-Adj. Close**	**B-Adj Close**	**Diff.**
1/3/2012	117925	78.37	370	5/1/2009	91600	59.44	2440
12/1/2011	114755	76.3	305	4/1/2009	94000	61.3	2050
11/1/2011	118500	78.76	360	3/2/2009	86700	56.4	2100
10/3/2011	116950	77.86	160	2/2/2009	78600	51.28	1680
9/1/2011	106800	71.04	240	1/2/2009	89502	59.78	-168
8/1/2011	109769	73	269	12/1/2008	96600	64.28	180
7/1/2011	111500	74.17	245	11/3/2008	104000	69.98	-970

6/1/2011	116105	77.39	20	10/1/2008	115490	76.8	290
5/2/2011	118775	79.07	170	9/2/2008	130600	87.9	-1250
4/1/2011	124750	83.3	-200	8/1/2008	116600	78.04	-460
3/1/2011	125300	83.63	-145	7/1/2008	114450	76.58	-420
2/1/2011	131300	87.28	380	6/2/2008	120750	80.24	390
1/3/2011	122425	81.75	-200	5/1/2008	134650	89.96	-290
12/1/2010	120450	80.11	285	4/1/2008	133850	89.14	140
11/1/2010	120200	79.68	680	3/3/2008	133400	89.46	-790
10/1/2010	119300	79.56	-40	2/1/2008	140000	93.49	-235
9/1/2010	124500	82.68	480	1/2/2008	136000	91	-500
8/2/2010	118675	78.78	505	12/3/2007	141600	94.72	-480
7/1/2010	117000	78.12	-180	11/1/2007	140100	93.8	-600
6/1/2010	120000	79.69	465	10/1/2007	132500	88.28	80
5/3/2010	105910	70.55	85	9/4/2007	118510	79.04	-50
4/1/2010	115325	77	-175	8/1/2007	118390	77.8	1690
3/1/2010	121800	81.27	-105	7/2/2007	110000	72.08	1880
2/1/2010	119800	80.13	-395	6/1/2007	109475	72.1	1325
1/4/2010	114600	76.43	-45	5/1/2007	109490	72.5	740
12/1/2009	99200	65.72	620	4/2/2007	109200	72.56	360
11/2/2009	100600	67.06	10	3/1/2007	108990	72.8	-210
10/1/2009	99000	65.66	510	2/1/2007	106190	70.46	500
9/1/2009	101000	66.46	1310	1/3/2007	110050	73.35	25
8/3/2009	100850	65.72	2270				μ = 367.31
7/1/2009	97000	63.61	1585				
6/1/2009	90000	57.92	3120				

Table 1: Berkshire-Hathaway A and B Share Prices and Differences

Table 1 lists beginning-of-month prices of Class A and B shares of Berkshire Hathaway stock for the 5-year period from year-end 2006 to January 2012. Split-adjusted closing prices are given for the first trading day of each month for both Class A and B shares. Based on dividend claims, each A share should sell for 1,500 times the B share price. The premium or difference between the A share prices and B share prices are given as *Diff* = (*A-Adj. Close)* - (1,500 ×*B-Adj. Close*). Thus, for example, on January 3, 2012, the adjusted closing price for A shares was

$117,925 and for B shares was $78.37. The difference between the A-shares price and B-shares price times 1,500 was *Diff* = $117,925 - (1,500 × $78.37) = $370. This means that a single A-share was worth $370 more than 1,500 B-shares. This difference had changed by 65 from December 1, 2011, when the difference was $305.

The mean difference over the five-year period was μ = $367.31. While, in a perfectly efficient market, one might expect for the mean difference to be zero for such an arbitrage portfolio, there may well be good reasons for the difference to tend to be positive. For example, the votes on the A-shares might justify the $367.31 average difference. Perhaps transactions costs might justify this difference. Regardless, this difference implies that the A shares tend to trade at a long-term mean premium of $367.31 over a portfolio of 1,500 B shares. We might anticipate that deviations from this long-run pairs relationship will tend to revert back to the mean. Figure 1 does seem to suggest that this pairs relationship does vary over time, but does tend to revert back to its mean.

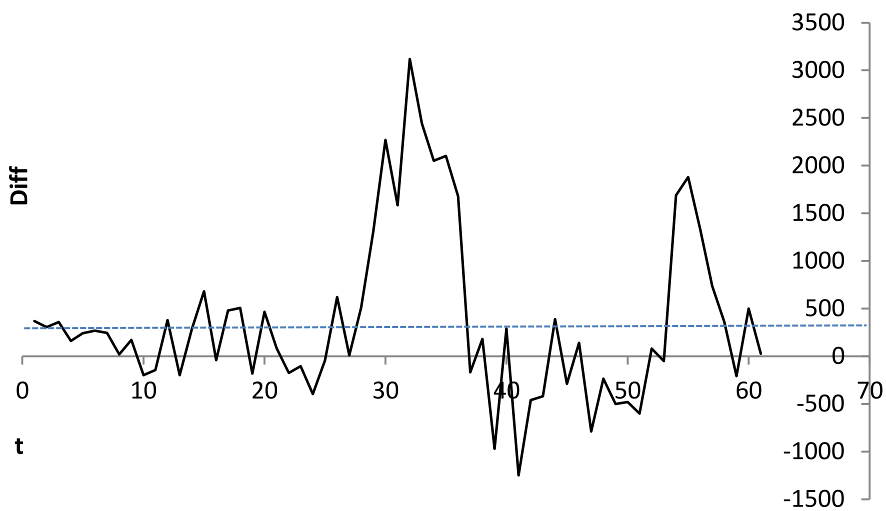

Figure 1: Arbitrage Portfolio Values over Time (t=0 is January 2007; t = 61 is January, 2012)

As we mentioned earlier, there are a variety of techniques for estimating pullback factors for pairs trading. The OLS regression and maximum likelihood methods are fairly straightforward statistical approaches, but are subject to a number of biases. Autoregressive (e.g., AR(1)) models are also used, but discussions of all of these techniques are beyond the scope of this book (See Yu [2009] for more discussion on calibration techniques). We might apply the ordinary least squares regression procedure to calibrate the Ornstein-Uhlenbeck-based pairs-trading model to estimate this difference:

$$d(Diff_t) = \lambda\left(\overline{Diff} - Diff_t\right)dt + \sigma_{Diff}dZ_t.$$

We obtained a long term mean of $\overline{Diff} = 367.31$, a pullback factor value of λ=.297 and a standard deviation σ_{Diff} = 883.4 so that:

$$d(Diff_t) = .297(367.31 - Diff_t)dt + 883.4dZ_t.$$

This model is intended to merely hint at an estimation technique, though, again, Yu [2009] can provide more input on this and other calibration procedures. This solution predicts that, the process will tend towards reverting back to its mean of $\overline{Diff} = 367.31$. If a horizontal line at the Diff value of 367.41 is drawn in Figure 6, we see that $Diff_t$ values tend to fluctuate randomly about the mean value. Even a large jolt that pushes the pairs return far away from the mean is followed by a reversion back to the mean.

Illustration: Stochastic Volatility

One of the key assumptions of the Black-Scholes options pricing model is that the underlying security volatility is constant over the life of the option. However, volatility is unobservable, yet it is clear to every derivatives trader that stock variances are not constant over time (e.g., the volatility clustering observed by Mandelbrot [1963]). In addition, variances are estimated with error (see, for example, (Muirhead [1987]), option pricing model valuations might be enhanced with the introduction of stochastic volatility parameters (e.g., Heston [1993]) and stochastic volatility models might help resolve empirical biases and anomalies such as the "smile effect" discussed in Chapter 7 (e.g., see Rubinstein [1985]). Stochastic volatility option pricing models include mean reversion and other time-varying processes for volatility, including Hull and White [1987], Stein and Stein [1991] and the square root process (similar to that in Cox, Ingersoll and Ross) of Heston [1993]. Heston's model takes the following form:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dZ_{S,t}$$

$$d\sigma_t^2 = \lambda(\bar{\sigma}^2 - \sigma_t^2)dt + \vartheta\sigma_t dZ_{\sigma,t},$$

where $\bar{\sigma}^2$ is the long-term mean variance, α is a constant and λ is the rate of reversion of the short-term variance σ_t^2 to the long-term mean. The term ϑ is the volatility of the volatility. In the equation below, ρ denotes the correlation coefficient between $Z_{S,t}$ and $Z_{\sigma,t}$. Denote the short term time-varying variance with $v(t) = \sigma_t^2$. Heston uses the standard no-arbitrage argument to derive a multivariate partial differential equation for valuing an asset with price P under the two sources of risk:

$$\frac{1}{2}vS^2\frac{\partial^2 P}{\partial S^2} + \frac{1}{2}\vartheta^2 v\frac{\partial^2 P}{\partial v^2} + \rho\vartheta vS\frac{\partial^2 P}{\partial S\partial v} + rS\frac{\partial P}{\partial S} + \{\lambda(\bar{\sigma}^2 - v) - \Theta v\}\frac{\partial P}{\partial v} + \frac{\partial P}{\partial t} = rP.$$

Heston offers a plausibility argument for choosing a risk premium parameter Θv to be proportional to the short term variance. To price a European call, the boundary conditions are difficult to deal with directly. So, Heston expresses the call in the form:

$$c_t = SP_1 - Xe^{-r(T-t)}P_2$$

where X is the exercise price of the call. The functions P_1 and P_2 both satisfy the partial differential equation above, and have simple boundary conditions. The solutions for P_1 and P_2 are obtained in integral form, and thus an integral representation is obtained for the solution. These integrals cannot be evaluated in closed form, but can be approximated numerically. While we invite the reader to explore the full derivation and computational techniques of Heston [1993], it is beyond the scope of this text to do so here.

There are a variety of factors that can lead to stochastic volatility for equities, including time-varying levels of uncertainty in the economy, uncertainty regarding corporate announcements, company leverage and institutional leverage affecting trading as equity prices decline. As we discussed earlier, these stochastic volatility models can be used for instruments related to FX, commodities and inflation rare-related instruments, etc. For example, as with interest rate securities, central bank rate management can lead to mean reverting volatilities as governments intervene in FX markets to stabilize rates.

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